“Structural Elimination of a Touchard Branch in Odd Perfect Numbers”

—Dual 3-adic and Exponent Constraints for $p = 13$, $\alpha = 5$—

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Abstract

We establish a complete structural elimination of the Touchard branch defined by $N = 13^5 m^2$ in the Eulerian form of an odd perfect number.
The argument combines the classical 3-adic valuation constraint imposed by the Touchard condition with a modular exponent restriction previously established in RP14.

We show that the condition $v_3(\sigma(m^2)) = 1$ forces the existence of a prime divisor $q \mid m$ with exponent $\beta \equiv 1 \pmod 3$, while RP14 proves that no such exponent can occur for primes $q \equiv 1 \pmod 3$ when 3 does not divide N.
This incompatibility yields an immediate contradiction, independent of cyclotomic or computational arguments.

The proof is entirely theoretical and relies solely on valuation theory and structural constraints.
As a consequence, no odd perfect number of the form $N = 13^5 m^2$ can exist.

1. Introduction

An odd perfect number, if it exists, must be of Eulerian form

$N = p^\alpha m^2$,

where p is an odd prime, $p \equiv \alpha \equiv 1 \pmod 4$, and $\gcd(p, m) = 1$.
Among the classical restrictions on such numbers, Touchard showed that either

$3 \mid N$
or
$3 \nmid N$ and $N \equiv 1 \pmod{12}$.

These two cases induce fundamentally different valuation constraints.
In recent work (RP14), structural restrictions on the exponents of primes dividing m were established in the branch $3 \nmid N$, based on congruence considerations.

The purpose of the present paper is to apply these restrictions to the specific Touchard branch defined by

$N = 13^5 m^2$,

and to show that this branch is structurally impossible.

2. Preliminaries

For a positive integer n, let $\sigma(n)$ denote the sum of its positive divisors, and for a prime p let $v_p(n)$ denote the p-adic valuation of n.

We recall the identity

$\sigma(q^{2\beta}) = (q^{2\beta+1} - 1)/(q - 1)$

for any prime q and integer $\beta \geq 1$.

We also use the Lifting the Exponent Lemma in its standard form for odd primes.

3. Forced Prime Divisors

Let

$N = 13^5 m^2$

be an odd perfect number. Then

$\sigma(N) = \sigma(13^5) \cdot \sigma(m^2) = 2N.$

A direct computation gives

$\sigma(13^5) = 1 + 13 + 13^2 + 13^3 + 13^4 + 13^5$
$= 2 \cdot 3 \cdot 7 \cdot 61 \cdot 157.$

Since $\gcd(13, m) = 1$, every odd prime divisor of $\sigma(13^5)$ must divide m.
Thus

$7 \mid m, \quad 61 \mid m, \quad 157 \mid m.$

No further assumptions on the prime factorization of m are made.

4. Dual Valuation Constraints and Immediate Contradiction

In this section we derive a contradiction using the 3-adic valuation constraint imposed by the Touchard condition, combined with the exponent restriction established in RP14. This argument is independent of any cyclotomic analysis.

4.1 Touchard 3-adic Constraint

Since we are in the Touchard branch with $3 \nmid N$, the classical Touchard condition yields

$v_3(\sigma(m^2)) = 1. \quad (4.1)$

4.2 3-adic Contribution of Prime Power Divisors

Let $q \equiv 1 \pmod 3$ be an odd prime and suppose $q^{2\beta} \parallel m^2$. Then

$\sigma(q^{2\beta}) = (q^{2\beta+1} - 1)/(q - 1).$

Since $q \equiv 1 \pmod 3$, the Lifting the Exponent Lemma gives

$v_3(q^{2\beta+1} - 1) = v_3(q - 1) + v_3(2\beta + 1).$

Hence

$$v_3(\sigma(q^{2\beta})) = v_3(2\beta + 1). \quad (4.2)$$

For the forced primes we compute

$$7 - 1 = 6 = 2 \cdot 3,$$
$$61 - 1 = 60 = 4 \cdot 3 \cdot 5,$$
$$157 - 1 = 156 = 4 \cdot 3 \cdot 13,$$

so that in each case

$$q \equiv 1 \pmod 3 \quad \text{and} \quad v_3(q - 1) = 1.$$

Therefore formula (4.2) applies uniformly to all forced divisors $q \in \{7, 61, 157\}$.

4.3 Global Exponent Constraint

Write

$$m = \prod q_i^{\beta_i},$$

where each $q_i \equiv 1 \pmod 3$. Then

$$v_3(\sigma(m^2))$$
$$= \sum v_3(\sigma(q_i^{2\beta_i}))$$
$$= \sum v_3(2\beta_i + 1). \quad (4.3)$$

By (4.1), the left-hand side equals 1.
Since each summand is a nonnegative integer, equation (4.3) implies that:

- Exactly one index i satisfies $v_3(2\beta_i + 1) = 1$,
- All other indices j satisfy $v_3(2\beta_j + 1) = 0$.

In particular, there exists a prime divisor $q \mid m$ such that

$$2\beta + 1 \equiv 0 \pmod 3,$$
or equivalently,

$$\beta \equiv 1 \pmod 3). \quad (4.4)$$

4.4 Conflict with RP14

RP14 establishes the following restriction:

If $3 \nmid N$ and $q \equiv 1 \pmod 3$ divides m, then
$\beta \not\equiv 1 \pmod 3$.

This restriction applies to every prime divisor of m in the present setting, since all forced divisors satisfy $q \equiv 1 \pmod 3$.

Condition (4.4) therefore contradicts the RP14 exponent restriction.

4.5 Elimination of the Branch p = 13, α = 5

The contradiction arises solely from the incompatibility between:

1. the Touchard constraint $v_3(\sigma(m^2)) = 1$, and
2. the RP14 restriction on exponents of primes $q \equiv 1 \pmod 3$.

No assumptions were made about additional prime divisors of m, and no computational enumeration was used.

We conclude that an odd perfect number of the form

$N = 13^5 m^2$

cannot exist.

5. Conclusion

We have shown that the Touchard branch defined by $p = 13$ and $\alpha = 5$ is structurally impossible.
The elimination follows directly from valuation-theoretic constraints and previously established exponent restrictions, without reliance on cyclotomic forcing or computation.

This result illustrates the effectiveness of combining local valuation constraints with global structural conditions in the study of odd perfect numbers.

Notes on Scope and Interpretation

The present work provides an unconditional elimination of the Touchard branch $p = 13$, $\alpha = 5$.
No claims are made regarding other Eulerian parameters or a complete resolution of the odd perfect number problem.
All arguments are purely structural and theoretical.